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# Some Applications of the Robin Function to Multivariable Approximation Theory

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Let $E \subset \mathbb{C}^n$ be compact, regular, and polynomially convex with pluricomplex Green function $V_E$. Given a sequence of polynomials $\{p_j\}_{j=1,2,\ldots}$, the first result is a condition for $(\overline{\lim_{j\to\infty}}\,(\log|p_j(z)|/\deg(p_j)))^*$ to equal $V_E$ on $\mathbb{C}^n - E$. The condition involves the Robin function of $E$ and the highest order homogeneous terms of the $p_j$ and generalizes one-variable results of Blatt–Saff. A second result gives a necessary and sufficient condition for $f$, the uniform limit of polynomials on $E$, to extend holomorphically to $E_R = \{z \mid V_E(z) < \log R\}$ for $R > 1$. The condition involves highest order homogeneous terms of best approximating polynomials to $f$ and the Robin function of $E$ and extends results of Szczepański. 

## 1. INTRODUCTION

Let $E \subset \mathbb{C}$ be compact with a connected complement. Let $G_E(z)$ denote the Green function of the exterior of $E$ with logarithmic singularity at $\infty$. We assume $E$ is regular (in the sense of potential theory). This means that, if we extend $G_E(z)$ to a function defined in the entire plane by setting $G_E(z) = 0$ for $z \in E$, then we obtain a function (still denoted by $G_E$) which is continuous on $\mathbb{C}$. In particular, the logarithmic capacity of $E$, denoted cap($E$), is non-zero. Let $W(E)$ denote the closure, in the uniform norm on $E$, of functions analytic on a neighborhood of $E$.

In their paper [2], Blatt and Saff prove the following results.

LEMMA 1.1 [2, Lemma 4.2]. *Let $\{p_d(z)\}_{d=1,2,\ldots}$ be a sequence of polynomials $p_d(z) = a_d z^d + \cdots$ (lower order terms), $a_d \neq 0$, satisfying*

$$\lim_{d\to\infty} \|p_d\|_E^{1/d} = 1 \tag{1.1}$$

$$\lim_{d\to\infty} |a_d|^{1/d} = \frac{1}{\operatorname{cap}(E)}. \tag{1.2}$$

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*Then $G_E(z)$ is an "exact harmonic majorant" (for definition see [2]) for $\{p_d(z)\}$ on $\mathbb{C} - E$.*

THEOREM 1.1 [2, Theorem 2.1]. *Let $f \in W(E)$ and, for each $d = 1, 2, \ldots$, let $B_d(z) = b_d z^d + \cdots + (\text{lower order terms})$ be the best approximant to $f$ (in the uniform norm) from $\mathcal{P}_d^1$ (the space of polynomials of degree $\leqslant d$ in one variable). Then $f$ is not analytic on $E$ if and only if*

$$\overline{\lim_{d \to \infty}} |b_d|^{1/d} = \frac{1}{\operatorname{cap}(E)}. \tag{1.3}$$

A related result of Wójcik [17] is

Theorem 1.2 [17, Theorem 3]. *Under the hypothesis of Theorem 1.1, $f$ has an analytic extension to $E_R$ for some $R > 1$ if and only if*

$$\overline{\lim_{d \to \infty}} |b_d|^{1/d} \leqslant \frac{1}{R \operatorname{cap}(E)}. \tag{1.4}$$

Here $E_R = \{z \in \mathbb{C} \mid G_E(z) < \log R\}$.

In this paper we will give multivariable versions of these results.

Let $E \subset \mathbb{C}^n$ be compact, polynomially convex, and regular (in the sense of pluripotential theory) with pluricomplex Green function $V_E(z)$. One definition of $V_E(z)$ is

$$V_E(z) = \sup_{\|p\|_E \leqslant 1} \left\{ \frac{1}{\deg(p)} \log |p(z)| \right\} \tag{1.5}$$

where the sup is over *all* polynomials $p$. (For the equivalence of this definition and (2.6) see [12].)

Theorem 2.1 and Corollary 2.1 show that $V_E(z)$ may be obtained as a pointwise upper envelope as in (1.5) or as a lim sup but using a subset of all polynomials. The subset is characterized by the homogeneous terms of highest order of the polynomials and the Robin function of $E$ (see (2.15)). This generalizes condition (1.2) of Lemma 1.1.

In one complex variable, the Tchebyshev polynomials (suitably normalized) satisfy the conditions of Lemma 1.1. Theorems 2.2 and 2.3 use Theorem 2.1 to give certain generalizations of this to the case of several variables. (See also Example 2.1.)

The proof of Theorem 2.1 relies on results in pluripotential theory due to Bedford and Taylor [3].

Theorem 3.1 gives necessary and sufficient conditions for $f \in W(E)$ (defined as in the one variable case) to be analytic on $E$ (i.e., on a neighborhood of $E$) or, given $R > 1$, to have an analytic extension to $E_R = \{z \in \mathbb{C}^n \mid V_E(z) < \log R\}$. Both (3.5) and Corollary 3.1 involve a condition for every point in $\mathbb{P}^{n-1}$, i.e., for each complex line through $0 \in \mathbb{C}^n$. These results generalize Theorems 1.1 and 1.2 to the multivariable setting. Theorem 3.2 gives an estimate for Tchebyshev polynomials in several variables and is of independent interest. An alternate proof of Theorem 3.2 has been given by Siciak [13].

Szczepański [14] has given necessary and sufficient conditions for $f$ to extend holomorphically to $E_R$. We show that conditions (3.4) and (3.5) of Theorem 3.1 imply the results of Szczepański. An explicit example where (3.4) is sharper than the result of Szczepański is given in Example 3.1.

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## 2. THE PLURICOMPLEX GREEN FUNCTION

We first recall some notation and terminology used in pluripotential theory (a general reference is the book of Klimek [5]). Let $\mathscr{L}$ denote the plurisubharmonic (p.s.h.) functions on $\mathbb{C}^n$ of logarithmic growth

$$\mathscr{L} = \{u \mid u \text{ is p.s.h. on } \mathbb{C}^n \text{ and } u(z) \leqslant \log^+ |z| + C\} \tag{2.1}$$

where $|z| = (\sum_{i=1}^n |z_i|^2)^{1/2}$ and $\log^+ |z| = \operatorname{Max}(0, \log |z|)$. Let

$$\mathscr{L}_+ = \{u \mid u \text{ is p.s.h. on } \mathbb{C}^n \text{ and } \log^+ |z| - C \leqslant u \leqslant \log^+ |z| + C\}. \tag{2.2}$$

The constants $C$ in (2.1) and (2.2) may depend on $u$.

For $u \in \mathscr{L}$ we define

$$\rho_u(z) = \varlimsup_{\substack{|\lambda| \to +\infty \\ \lambda \in \mathbb{C}}} \{u(\lambda z) - \log^+ |\lambda z|\} \tag{2.3}$$

and

$$\bar{\rho}_u(z) = \varlimsup_{\substack{|\lambda| \to +\infty \\ \lambda \in \mathbb{C}}} \{u(\lambda z) - \log |\lambda|\}. \tag{2.4}$$

Following the convention used by Bedford and Taylor [3] we consider $\rho_u$ to be defined on $\mathbb{P}^{n-1}$ (complex projective $(n-1)$-space) and refer to it as the Robin function of $u$. (In the literature, the function $\bar{\rho}_u(z)$ is sometimes called the Robin function of $u$). We let $[z]$ denote the point in $\mathbb{P}^{n-1}$

determined by $z \in \mathbb{C}^n - \{0\}$, and we will use the notation $\rho_u([z])$ for the value of the Robin function at this point. Clearly,

$$\rho_u([z]) = \bar{\rho}_u(z) - \log |z| \qquad \text{for} \quad z \in \mathbb{C}^n - \{0\}. \tag{2.5}$$

Let $E \subset \mathbb{C}^n$ be a compact set. Its pluricomplex Green function, denoted by $V_E$, is defined by

$$V_E(z) = \sup \{u(z) \mid u \in \mathscr{L} \text{ and } u \leqslant 0 \text{ on } E\}. \tag{2.6}$$

$E$ is said to be *regular* if $V_E$ is continuous on $\mathbb{C}^n$. For $E$ regular the Robin function of $E$, denoted by $\rho_E$, is defined to be the Robin function of $V_E$. If $E$ is polynomially convex and regular then $V_E(z) = 0$ if and only if $z \in E$.

PROPOSITION 2.1. (i) *Let $u \in \mathscr{L}$. Then $\bar{\rho}_u$ is p.s.h. on $\mathbb{C}^n$.*

(ii) *Let $u \in \mathscr{L}$. Then $\rho_u$ is upper semi-continuous (u.s.c.) on $\mathbb{P}^{n-1}$.*

(iii) *Let $u, v \in \mathscr{L}$. Suppose that $\rho_u = \rho_v$ almost everywhere (a.e.) on $\mathbb{P}^{n-1}$. Then $\rho_u = \rho_v$ at all points of $\mathbb{P}^{n-1}$.*

(iv) *For $E$ regular, $\rho_E$ is continuous on $\mathbb{P}^{n-1}$.*

*Proof.* (i) (see also [13], [18]) We define the function

$$\tilde{u}(t, z) = \log |t| + u(z/t) \qquad \text{for} \quad (t, z) \in \mathbb{C} \times \mathbb{C}^n \quad \text{with } t \neq 0 \tag{2.7}$$

and

$$\tilde{u}(0, z) = \varlimsup_{\substack{(t, \xi) \to (0, z) \\ t \neq 0}} \tilde{u}(t, \xi). \tag{2.8}$$

Then $\tilde{u}$ is p.s.h. on $\mathbb{C} \times \mathbb{C}^n$ by [5, Theorem 2.9.22].

Using [4, Proposition 5.1] one deduces that

$$\tilde{u}(0, z) = \varlimsup_{\substack{(t, z) \to (0, z) \\ t \neq 0}} \tilde{u}(t, z) \tag{2.9}$$

and so

$$\bar{\rho}_u(z) = \tilde{u}(0, z) \qquad \text{for} \quad z \in \mathbb{C} \tag{2.10}$$

and (i) follows.

(ii) follows from (i) and (2.5). (iii) follows from (i), (2.5), and [5, Corollary 2.9.8]. (iv) is a result of Levenberg [6] (also proven in [13]). ∎

For $u$ a locally bounded p.s.h. function on $\mathbb{C}^n$, $(dd^c u)^n$ denotes the Monge–Ampère operator on $u$. This is locally finite positive Borel measure and for $u \in \mathcal{L}_+$ it is a finite measure on $\mathbb{C}^n$ (see [5, 15]). $(dd^c V_E)^n$ has support in $E$.

Lemma 2.1 below is similar to results in [3].

LEMMA 2.1. *Let $E \subset \mathbb{C}^n$ be compact, regular, and polynomially convex. Let $v \in \mathcal{L}$ satisfy $v(z) \leqslant V_E(z)$ for all $z \in \mathbb{C}^n$. Suppose that $\rho_v = \rho_E$ for all $[z] \in \mathbb{P}^{n-1}$. Then $v(z) = V_E(z)$ for all $z \in \mathbb{C}^n - E$.*

*Proof.* Choose $c$ so that $\log|z| - c < 0$ on $E$. Let $w = \mathrm{Max}(v, 0, \log|z| - c)$. Then $w \in \mathcal{L}_+$, $\rho_w = \rho_E$ and $w(z) \leqslant V_E(z)$ for all $z \in \mathbb{C}^n$. By Theorem 6.1 of [3] we have

$$\int_{\mathbb{C}^n} V_E(dd^c w)^n \leqslant \int_{\mathbb{C}^n} w(dd^c V_E)^n. \tag{2.11}$$

Now, the right-hand side of (2.11) is zero since $\mathrm{supp}(dd^c V_E)^n \subseteq E$. Hence, since the left-hand side of (2.11) is non-negative, we have

$$\int_{\mathbb{C}^n} V_E(dd^c w)^n = 0. \tag{2.12}$$

Since $V_E(z) > 0$ for all $z \in \mathbb{C}^n - E$, we conclude that $\mathbb{C}^n - E$ is a $(dd^c w)^n$ set of measure zero. Thus $V_E \leqslant w$ for $(dd^c w)^n$ almost all points in $\mathrm{supp}(dd^c w)^n$. Now using Lemma 6.5 of [3] we conclude that $V_E(z) \leqslant w(z)$ for all $z \in \mathbb{C}^n$. Thus $V_E(z) = w(z)$ for all $z \in \mathbb{C}^n$ and since $V_E(z) > 0$ and $V_E(z) > \log|z| - c$ for $z \in \mathbb{C}^n - E$ we have $V_E(z) = v(z)$ for $z \in \mathbb{C}^n - E$. ∎

For $p$ a polynomial of degree $d$ in $n$ variables we let $\hat{p}(z)$ denote the homogeneous polynomial, sum of terms of degree $d$. That is, for $p = \sum_{|\alpha| \leqslant d} a_\alpha z^\alpha$ with $a_\alpha \neq 0$ for some $\alpha$, $|\alpha| = d$ then $\hat{p} = \sum_{|\alpha| = d} a_\alpha z^\alpha$. Condition (1.2) may be written as

$$\lim_{d \to \infty} \left( \frac{1}{d} \log|\hat{p}_d(z)| \right) - \log|z| = \rho_E \tag{2.13}$$

where $\rho_E = \log(1/\mathrm{cap}(E))$ is the Robin constant of $E$. Thus (2.15) is a multivariable version of (1.2).

We will use the following standard notation. Let $f$ be a function defined on an open set $H \subset \mathbb{C}^n$. We let $f^*$ denote its u.s.c. regularization. That is, for $z \in H$, $f^*(z) = \overline{\lim}_{\xi \to z} f(\xi)$.

THEOREM 2.1. *Let $E \subset \mathbb{C}^n$ be compact, regular, and polynomially convex. Let $\{p_j\}_{j=1,2,\ldots}$, be a sequence of polynomials satisfying*

$$\overline{\lim_{j\to\infty}} \|p_j\|_E^{1/\deg p_j} = 1, \tag{2.14}$$

$$\left(\overline{\lim_{j\to\infty}} \left\{\frac{1}{\deg(p_j)} \log |\hat{p}_j(z)|\right\}\right)^* - \log |z|$$

$$= \rho_E([z]) \qquad \text{for all} \quad z \in \mathbb{C}^n - \{0\}. \tag{2.15}$$

*Then*

$$\left(\overline{\lim_{j\to\infty}} \left\{\frac{1}{\deg(p_j)} \log |p_j(z)|\right\}\right)^* = V_E(z) \qquad \text{for all} \quad z \in \mathbb{C}^n - E.$$

*Proof.* Let $v = (\overline{\lim}_{j\to\infty}\{(1/\deg(p_j)) \log |p_j(z)|\})^*$. Then $v$ is plurisubharmonic on $\mathbb{C}^n$ [5, Prop. 2.9.17]. We will show that $v$ satisfies the hypothesis of Lemma 2.1 and then Theorem 2.1 will follow. First, from (2.14) we have that for every $\varepsilon > 0$ there exists $j_0$ such that

$$\frac{1}{\deg(p_j)} \log |p_j(z)| \leqslant \varepsilon \qquad \text{on } E \quad \text{for } j \geqslant j_0.$$

Hence $(1/\deg(p_j)) \log |p_j(z)| \leqslant \varepsilon + V_E(z)$ on $\mathbb{C}^n$ for $j \geqslant j_0$ and we conclude that $v(z) \leqslant V_E(z)$ for all $z \in \mathbb{C}^n$.

This implies that $\rho_v \leqslant \rho_E$ for all $[z] \in \mathbb{P}^{n-1}$ and so to show that $\rho_v = \rho_E$ for all $[z] \in \mathbb{P}^{n_1}$ we need only to show that $\rho_v \geqslant \rho_E$ for all $[z] \in \mathbb{P}^{n-1}$. We will use a method of Zeriahi [18].

For $u$ be subharmonic on $\mathbb{C}$ and in the class $\mathscr{L}$, the log convexity of $\mathrm{Max}_{|\lambda|=r}\, u(\lambda)$ implies that

$$\overline{\lim_{|\lambda|\to\infty}}\; u(\lambda) - \log |\lambda| = \inf_{r\geqslant 1}(\mathop{\mathrm{Max}}_{|\lambda|=r} u(\lambda) - \log r). \tag{2.16}$$

Fix $z \in \mathbb{C}^n - \{0\}$ and apply (2.16) to the function of the single complex variable $\lambda$,

$$\lambda \to \frac{1}{\deg(p_j)} \log |p_j(\lambda z)|.$$

We obtain

$$\frac{\log |\hat{p}_j(z)|}{\deg(p_j)} \leqslant \mathop{\mathrm{Max}}_{|\lambda|=r} \left(\frac{\log |p_j(\lambda z)|}{\deg(p_j)} - \log r\right). \tag{2.17}$$

Taking $\overline{\lim}$ of both sides, and using Hartogs' Lemma [5, Theorem 2.6.4] on the right-hand side of (2.17) we have

$$\overline{\lim_{j\to\infty}} \frac{\log |\hat{p}_j(z)|}{\deg(p_j)} \leqslant \operatorname*{Max}_{|\lambda|=r} \overline{\lim_{j\to\infty}} \left(\frac{\log |p_j(\lambda z)|}{\deg(p_j)} - \log r\right). \tag{2.18}$$

The right side of (2.18) is

$$\leqslant \operatorname*{Max}_{|\lambda|=r} (v(\lambda z) - \log r).$$

Letting $r \to \infty$, we have

$$\overline{\lim_{j\to\infty}} \frac{\log |\hat{p}_j(z)|}{\deg(p_j)} - \log |z| \leqslant \rho_v([z]).$$

Using (2.15) and the fact the $\rho_v$ is u.s.c. on $\mathbb{P}^{n-1}$ we conclude that $\rho_E([z]) \leqslant \rho_v([z])$ for all $[z] \in \mathbb{P}^{n-1}$.

COROLLARY 2.1. *Let $\{p_j\}_{j=1,2,\ldots}$ be a sequence of polynomials satisfying*

$$\|p_j\|_E \leqslant 1, \tag{2.19}$$

$$\left(\sup_j \left(\frac{1}{\deg(p_j)} \log |\hat{p}_j(z)|\right)\right)^* - \log |z| = \rho_E([z]) \qquad \text{for all} \quad z \in \mathbb{C}^n - \{0\}. \tag{2.20}$$

*Then*

$$\left(\sup_j \left(\frac{1}{\deg(p_j)} \log |p_j(z)|\right)\right)^* = V_E(z) \qquad \text{for all} \quad z \in \mathbb{C}^n - E.$$

*Proof.* This can be deduced from Lemma 2.1 in a similar fashion to Theorem 2.1. ∎

*Remark* 2.1. Using Proposition 2.1(iii) the equality in (2.15) may be replaced by equality a.e. in $\mathbb{P}^{n-1}$ and the conclusion of Theorem 2.1 is still valid. The case is similar for (2.20) and Corollary 2.1.

Given a homogeneous polynomial $H(z)$ of degree $d$ we denote by $\mathrm{Tch}_E(H)$ a Tchebyshev polynomial for $E$ with leading term $H$. That is, $\mathrm{Tch}_E(H) = H + h$ where $h$ is a polynomial of deg $\leqslant d-1$ and $\|H+h\|_E \leqslant \|H+h_1\|_E$ for all polynomials $h_1$ of degree $\leqslant d-1$.

Of course, $\mathrm{Tch}_E(H)$ is, in general, not unique, but, since the norms of any Tchebyshev polynomials are all the same, $\|\mathrm{Tch}_E(H)\|_E$ is unambiguously defined.

Let $\mu$ be a finite Borel measure on $E$. $\mu$ is said to satisfy the Bernstein–Markov inequality if for all $\varepsilon > 0$ there is a constant $C = C(\varepsilon) > 0$ such that, for all (analytic) polynomials $P$,

$$\|P\|_E \leqslant C(1+\varepsilon)^{\deg(P)} \, \|P\|_{L^2(E,\mu)}.$$

It is known that the equilibrium measure on $E$ satisfies the Bernstein–Markov inequality [16]. Since $E$ is regular, it is non-pluripolar (see [5]). Hence if $\|P\|_E = 0$ for an analytic polynomial $P$, then $P \equiv 0$. It then follows from the Bernstein–Markov inequality that the monomials are linearly independent in $L^2(E, \mu)$. We consider the $n$-multi-indices as ordered lexicographically and denote by $\{P_\alpha(z)\}$ the orthonormal polynomials obtained by using the Gram–Schmidt procedure on the monomials. Here $\alpha \in \mathbb{N}^n$ is an $n$ multi-index and $P_\alpha(z)$ is a linear combination of $z^\alpha$ and monomials of lower lexicographic order.

THEOREM 2.2. *Let $E \subset \mathbb{C}^n$ be compact, regular, and polynomially convex. Let $\mu$ be a finite positive Borel measure on E which satisfies the Bernstein–Markov inequality. Then*

$$\left( \overline{\lim_{\alpha}} \, \frac{1}{|\alpha|} \log |\mathrm{Tch}_E \, \hat{P}_\alpha(z)| \right)^* = V_E(z) \qquad \text{for all} \quad z \in \mathbb{C}^n - E.$$

*Proof.* It is a result of Zeriahi [18] that, under the above hypothesis $\overline{\lim}_\alpha \, (1/|\alpha|) \log |\hat{P}_\alpha(z)| - \log |z| = \rho_E([z])$ for all $z \in \mathbb{C}^n - \{0\}$. Furthermore, for all $\alpha$, $\|\mathrm{Tch}_E \, \hat{P}_\alpha(z)\|_E \leqslant \|P_\alpha(z)\|_E \leqslant C(1+\varepsilon)^{|\alpha|}$ where the rightmost inequality follows from the Bernstein–Markov inequality. Thus, the family of polynomials $\{\mathrm{Tch}_E \, \hat{P}_\alpha(z)\}_{\alpha \in \mathbb{N}^n}$ satisfies the hypothesis of Theorem 2.1 and Theorem 2.2 follows. ▮

Let $w \in \mathbb{C}^n$ with $|w| = 1$. Consider the constants

$$\kappa_d = \kappa_d(E, w) = \inf\{ \|p\|_E \mid p \text{ is a polynomial}, \; \deg(p) = d \text{ and } |\hat{p}(w)| = 1\}. \tag{2.21}$$

It is easy to see that $\kappa_{d+s} \leqslant \kappa_d \kappa_s$ for $d, s$ positive integers and we set [1, Corollary 4.9.20]

$$\kappa = \kappa(E, w) = \lim_{d \to \infty} (\kappa_d)^{1/d} = \inf_{d \geqslant 1} \kappa_d^{1/d}. \tag{2.22}$$

In the case of one complex variable, $\kappa$ is the Tchebyshev constant of $E$. The next proposition is a generalization of the classical relation between the Tchebyshev constant and the Robin constant. A proof has been given by S. Nivoche [7, Proposition 4.2]. See also [13].

PROPOSITION 2.2. *Let $E \subset \mathbb{C}^n$ be compact and regular. Then* $\kappa(E, w) = \exp(-\rho_E([w]))$.

Lemma 1.1 holds for the Tchebyshev polynomials suitably normalized. Example 2.1 shows that, in the several variable case, the family of polynomials consisting of Tchebyshev polynomials for all monomials does *not*, in general, have the multivariable property corresponding to the conclusion of Lemma 1.1 (i.e., with the pluricomplex Green function replacing the Green function). A positive result however is given by Theorem 2.3.

THEOREM 2.3. *Let $E \subset \mathbb{C}^n$ be compact, regular, and polynomially convex. Let $\alpha_1, \alpha_2, \ldots$ be a countable set of points in $\mathbb{C}^n$ such that $|\alpha_j| = 1$ for all $j$ and $\{[\alpha_j]\}_{j=1,2,\ldots}$ is dense in $\mathbb{P}^{n-1}$. For $d, j = 1, 2, \ldots$ let $Q_{d,j}$ be a polynomial satisfying* $\deg(Q_{d,j}) = d$, $\|Q_{d,j}\|_E \leqslant 1$, *and* $|\hat{Q}_{d,j}(\alpha_j)| = 1/\kappa_d(E, \alpha_j)$. *Then*

$$\left(\sup_{d,j}\left(\frac{1}{d}\log|Q_{d,j}(z)|\right)\right)^* = V_E(z) \qquad \text{for} \quad z \in \mathbb{C}^n - E.$$

*Proof.*

$$\left(\sup_{d,j}\left(\frac{1}{d}\log|\hat{Q}_{d,j}(z)|\right)\right)^* - \log|z| \leqslant \rho_E([z]) \tag{2.23}$$

for all $[z] \in \mathbb{P}^{n-1}$ and we have equality at $[\alpha_j]_{j=1,2,\ldots}$. Hence, since the left side of (2.23) is u.s.c. and the right side is continuous (by Proposition 2.1(iv)) we have equality in (2.23) at all points of $\mathbb{P}^{n-1}$ and Corollary 2.1 applies.

EXAMPLE 2.1. Let $E \subset \mathbb{C}$ be compact, regular, and polynomially convex. Then the family of polynomials $\{p_d(z)\}_{d=1,2,\ldots}$, where

$$p_d(z) = \frac{\mathrm{Tch}_E(z^d)}{\|\mathrm{Tch}_E(z^d)\|_E}, \tag{2.24}$$

satisfies the hypothesis of Lemma 1.1 so that

$$G_E(z) = \left(\sup_{d \geqslant 1}\frac{1}{d}\log|p_d(z)|\right)^* \qquad \text{for} \quad z \in \mathbb{C} - E.$$

We will give an example of a set $E \subset \mathbb{C}^2$ compact, regular, and polynomially convex for which given the family of polynomials $\{p_{m_1 m_2}(z_1, z_2)\}_{(m_1, m_2) \in \mathbb{N}^2}$ where

$$p_{m_1 m_2}(z_1, z_2) = \frac{\mathrm{Tch}_E(z_1^{m_1} z_2^{m_2})}{\|\mathrm{Tch}_E(z_1^{m_1} z_2^{m_2})\|_E} \tag{2.25}$$

we have

$$V_E(z_1, z_2) \neq \sup_{(m_1, m_2) \in \mathbb{N}^2} \frac{1}{m_1 + m_2} \log |p_{m_1 m_2}(z_1, z_2)|$$

at some points of $\mathbb{C}^2 - E$.

Let $0 < \gamma < 1$ and let

$$K_\gamma = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1| \leqslant 1, |z_2| \leqslant 1 \text{ and } |z_1 - z_2| \leqslant \gamma\}. \tag{2.26}$$

Then $K_\gamma$ is compact, and polynomially convex. It is regular by a result of Pleśniak [8]. $K_\gamma$ is invariant under $(z_1, z_2) \to (e^{i\theta} z_1, e^{i\theta} z_2)$. Hence, a Tchebyshev polynomial with leading term a given homogeneous polynomial is given by that polynomial as may be seen by averaging over $\theta$. Note that $\|z_1^{m_1} z_2^{m_2}\|_{K_\gamma} = 1$ since $(1, 1) \in K_\gamma$. Thus we may take $p_{m_1 m_2}(z_1, z_2) = z_1^{m_1} z_2^{m_2}$ for all $(m_1, m_2) \in \mathbb{N}^2$.

Now

$$\sup_{(m_1, m_2) \in \mathbb{N}^2} \frac{1}{m_1 + m_2} \log |z_1^{m_1} z_2^{m_2}| = \mathrm{Max}(\log |z_1|, \log |z_2|). \tag{2.27}$$

Also, from the definition of the pluricomplex Green function given in (1.5),

$$V_{K_\gamma}(z_1, z_2) \geqslant \mathrm{Max}(\log |z_1|, \log |z_2|, \log(|z_1 - z_2|/\gamma)). \tag{2.28}$$

It follows that $V_{K_\gamma}(1, 0) \geqslant -\log \gamma > 0$ whereas, evaluated at $(1, 0)$, the function on the right side of (2.27) has value 0.

## 3. POLYNOMIAL APPROXIMANTS

Let $E \subset \mathbb{C}^n$ be compact, polynomially convex, and regular. Let $W(E)$ denote the closure in the sup norm on $E$ of the functions holomorphic in a neighborhood of $E$. In the one-variable case, Mergelyan's theorem states that $W(E)$ coincides with the functions continuous on $E$ and holomorphic in its interior. We let, for $R>1$,

$$E_R = \{z \in \mathbb{C}^n \mid V_E(z) < \log R\}. \tag{3.1}$$

Let $f \in W(E)$ and let $B_d (d = 1, 2, ...)$ be a sequence of best approximants to $f$ from $\mathscr{P}^n_d$ (the space of polynomials of deg $\leqslant d$ in $n$ variables). Let $\beta_d$ denote the sum of terms in $B_d$, homogeneous of degree $d$. That is, if $\deg(B_d) < d$ then $\beta_d \equiv 0$ but if $\deg(B_d) = d$ then $\beta_d = \hat{B}_d$.

THEOREM 3.1. *Let $R>1$. Consider the following four properties*:

$$f \text{ extends holomorphically to } E_R, \tag{3.2}$$

$$\overline{\lim_{d\to\infty}} \|f - B_d\|_E^{1/d} \leqslant \frac{1}{R}, \tag{3.3}$$

$$\overline{\lim_{d\to\infty}} \|\mathrm{Tch}_E\, \beta_d\|_E^{1/d} \leqslant \frac{1}{R}, \tag{3.4}$$

$$\overline{\lim_{d\to\infty}} \frac{1}{d} \log |\beta_d(z)| - \log |z| \leqslant \rho_E([z]) - \log R \quad \text{for all} \quad z \in \mathbb{C}^n - \{0\}. \tag{3.5}$$

*Then* (3.2) $\Leftrightarrow$ (3.3) $\Leftrightarrow$ (3.4) $\Leftrightarrow$ (3.5).

*Proof.* That (3.2) and (3.3) are equivalent is a result of Siciak [12, Theorem 8.5] generalizing the one-variable results of Bernstein and Walsh.

We will first prove the equivalence of (3.3) and (3.4). The method of proof is the same as that of [2].

(3.3) $\Rightarrow$ (3.4). Let $\{B_d\}$ be a sequence of polynomials satisfying (3.3). Let $r$ satisfy $1<r<R$. Then there exists a positive integer $d_0$ such that for all $d \geqslant d_0$ we have $\|f - B_d\|_E \leqslant r^{-d}$. Now, for $d \geqslant d_0 + 1$

$$\|\mathrm{Tch}_E\, \beta_d\|_E \leqslant \|B_d - B_{d-1}\|_E \leqslant \|B_d - f\|_E + \|B_{d-1} - f\|_E \leqslant r^{-d}(1+r).$$

Hence $\overline{\lim}_d \|\mathrm{Tch}_E\, \beta_d\|_E^{1/d} \leqslant r^{-1}$ and since this holds for all $r<R$, (3.4) holds.

(3.4) $\Rightarrow$ (3.3). Let $r$ satisfy $1<r<R$. Then there exists a positive integer $d_0$, such that, for all $d \geqslant d_0$, we have

$$\|\mathrm{Tch}_E\, \beta_d\|_E \leqslant \frac{1}{r^d}.$$

Now, for $d \geqslant d_0$

$$\begin{aligned}\|f - B_d\|_E &\leqslant \|f - B_{d+1}\|_E + \|\mathrm{Tch}_E\, \beta_{d+1}\|_E \\ &= \|f - B_{d+1}\|_E + r^{-(d+1)}.\end{aligned}$$

Repeating the above reasoning on $\|f - B_{d+1}\|_E$, $\|f - B_{d+2}\|_E$, ... we get $\|f - B_d\|_E \leqslant r^{-d}/(r-1)$. Hence $\overline{\lim}_{d\to\infty} \|f - B_d\|_E^{1/d} \leqslant 1/r$ and since this holds for all $r < R$, (3.3) follows.

(3.4) $\Rightarrow$ (3.5). As above, let $r$ satisfy $1 < r < R$. There exists a positive integer $d_0$ such that for $d \geqslant d_0$, $\|\mathrm{Tch}_E\, \beta_d\|_E^{1/d} \leqslant 1/r$. That is,

$$\log r + \frac{1}{d} \log |\mathrm{Tch}_E\, \beta_d(z)| \leqslant 0 \qquad \text{on } E.$$

But $\mathrm{Tch}_E\, \beta_d(z)$ is a polynomial of deg $d$ (or identically zero) so the left-hand side in the above inequality is in the class $\mathscr{L}$ (or identically, $-\infty$). Thus

$$\log r + \frac{1}{d} \log |\mathrm{Tch}_E\, \beta_d(z)| \leqslant V_E(z) \qquad \text{for all} \quad z \in \mathbb{C}^n.$$

It follows that $\log r + (1/d) \log |\beta_d(z)| - \log |z| \leqslant \rho_E([z])$ for all $z \in \mathbb{C}^n - \{0\}$. Since this holds for all $d \geqslant d_0$ and all $r < R$, condition (3.5) follows.

For $n = 1$, $\rho_E(z) = \rho_E$, the Robin constant of $E$. For $B_d(z) = \beta_d z^d +$ (lower order terms), condition (1.4) of Theorem 1.2 and condition (3.5) of Theorem 3.2 are equivalent.

The fact that (3.5) $\Rightarrow$ (3.2) in the case $n = 1$ is due to Wójcik [17]. Theorem 3.2 below shows that (3.5) $\Rightarrow$ (3.4), in the case $n \geqslant 1$, by applying it to the polynomials $R^d \beta_d$ where $\{\beta_d\}$ satisfy (3.5).

For $n = 1$ all homogeneous polynomials are constant multiples of monomials. For $n > 1$, this is, of course, not the case and proving the estimates on Tchebyshev polynomials needed to show that (3.5) $\Rightarrow$ (3.4) is more complicated.

THEOREM 3.2. *Let $E$ be compact, regular, and polynomially convex. Let $\{\hat{Q}_d\}$ be a sequence of homogeneous polynomials with* $\deg(\hat{Q}_d) = d$ *(or $\hat{Q}_d \equiv 0$) for $d = 1, 2, 3, \ldots$. Suppose that*

$$\overline{\lim_{d\to\infty}} \frac{1}{d} \log |\hat{Q}_d(z)| - \log |z| \leqslant \rho_E([z]) \qquad \text{for all} \quad z \in \mathbb{C}^n - \{0\}. \tag{3.6}$$

*Then*

$$\overline{\lim_{d \to \infty}} \|\mathrm{Tch}_E \hat{Q}_d\|_E^{1/d} \leqslant 1. \tag{3.7}$$

The first step in the proof of Theorem 3.2 is Lemma 3.1.

For $f$ holomorphic on an open subset of $\mathbb{C}^n$ we will use the notation $df = \sum_{i=1}^n (\partial f/\partial z_i)\, dz_i$.

LEMMA 3.1. *Given* $\varepsilon > 0$, *there exist finitely many polynomials* $W_1, ..., W_s$ *satisfying* (3.8), (3.9), *and* (3.10) *below.*

$$\|W_j\|_E \leqslant 1 \qquad \text{for} \quad j = 1, ..., s \tag{3.8}$$

$$\begin{aligned} &\mathrm{Max}_{1 \leqslant j \leqslant s} \left( \frac{\log |\hat{W}_j(z)|}{\deg(W_j)} \right) - \log |z| \\ &\quad \geqslant \rho_E([z]) - \varepsilon \qquad \text{for all} \quad z \in \mathbb{C}^n - \{0\}. \end{aligned} \tag{3.9}$$

For any subset of cardinality $n$, $\{i_1, ..., i_n\} \subset \{1, ..., s\}$, then

$$d\hat{W}_{i_1} \wedge d\hat{W}_{i_2} \wedge \cdots \wedge d\hat{W}_{i_n} \not\equiv 0 \qquad \text{on } \mathbb{C}^n. \tag{3.10}$$

*Proof* (*of Lemma* 3.1). Let $B = \{z \in \mathbb{C}^n \mid |z| = 1\}$. Given $z_0 \in B$ and $\varepsilon > 0$, there exists, by Proposition 2.2, a polynomial $p$ such that $\|p\|_E \leqslant 1$ and

$$\frac{\log |\hat{p}(z_0)|}{\deg(p)} \geqslant \rho_E([z_0]) - \frac{\varepsilon}{2}. \tag{3.11}$$

Since $\rho_E$ is continuous on $\mathbb{P}^{n-1}$, there is a neighborhood $N$ of $z_0$ in $B$ such that

$$\frac{\log |\hat{p}(z)|}{\deg(p)} \geqslant \rho_E([z]) - \varepsilon \qquad \text{for all} \quad z \in \bar{N}$$

where $\bar{N}$ denotes the closure of $N$.

Finitely many such neighborhoods $N_1, ..., N_s$ cover $B$ and the associated polynomials $p_1, ..., p_s$ satisfy (3.8) and (3.9). We may also assume $p_1, ..., p_s$ are all of the same degree, say $D$, since (3.8) and (3.9) are unchanged if any of the polynomials is raised to a power. To obtain polynomials which also satisfy (3.10) we will first consider small perturbations of $p_1, ..., p_s$ given by

$$q_j(\eta_j, z) = p_j(z) + \eta_j \langle z, \alpha_j \rangle^D \qquad \text{for} \quad j = 1, ..., s \tag{3.12}$$

where $\eta = (\eta_1, ..., \eta_s) \in \mathbb{C}^s$ and $\alpha_j \in \mathbb{C}^n - \{0\}$. Here $\langle z, \alpha_j \rangle = \sum_{k=1}^n z_k \alpha_{jk}$.

The $\alpha_1, ..., \alpha_s$ are chosen so that any subset of $n$ of them is linearly independent. This implies that, for any subset of cardinality $n$, $\{i_1, ..., i_n\} \subset \{1, ..., s\}$, we have

$$d(\langle z, \alpha_{i_1}\rangle)^D \wedge \cdots \wedge d(\langle z, \alpha_{i_n}\rangle^D) \not\equiv 0 \qquad \text{on } \mathbb{C}^n. \tag{3.13}$$

We set

$$W_j(\eta_j, z) = \frac{q_j(\eta_j, z)}{\|q_j(\eta_j, z)\|_E} \qquad \text{for} \quad j = 1, ..., s. \tag{3.14}$$

We will show that there exists a point $\eta \in \mathbb{C}^s$ so that the corresponding $W_1, ..., W_s$ satisfy (3.8), (3.9), and (3.10) by showing that (3.8) and (3.9) are satisfied for all $\eta \in \mathbb{C}^s$ with $|\eta|$ sufficiently small and (3.10) is satisfied for all $\eta$ in a dense open set in $\mathbb{C}^s$. For $|\eta_j|$ sufficiently small, $\|q_j(\eta_j, z)\|_E \neq 0$, so $W_j$ satisfies (3.8).

Note that

$$\hat{q}_j(\eta_j, z) = \hat{p}_j(z) + \eta_j \langle z, \alpha_j\rangle^D. \tag{3.15}$$

Given $\varepsilon > 0$, for $|\eta_j|$ sufficiently small, it follows from (3.12) and (3.14) that

$$\frac{1}{D} \log |\hat{W}_j(\eta_j, z)| \geqslant \frac{1}{D} \log |\hat{q}_j(\eta_j, z)| - \varepsilon \qquad \text{for all} \quad z \in \bar{N}_j. \tag{3.16}$$

Also, from (3.15),

$$\frac{1}{D} \log |\hat{q}_j(\eta_j, z)| \geqslant \frac{1}{D} \log |\hat{p}_j(z)| - \varepsilon \qquad \text{for all} \quad z \in \bar{N}_j \tag{3.17}$$

and, from the definition of $p_j(z)$,

$$\frac{1}{D} \log |\hat{p}_j(z)| \geqslant \rho_E([z]) - \varepsilon \qquad \text{for all} \quad z \in \bar{N}_j. \tag{3.18}$$

Combining (3.16), (3.17), and (3.18) we see that $W_j(\eta_j, z)$ satisfies (3.9) (with $3\varepsilon$ replacing $\varepsilon$) for all $|\eta_j|$ sufficiently small.

For any subset of cardinality $n$, $\{i_1, ..., i_n\} \subset \{1, ..., s\}$, we consider $d\hat{q}_{i_1} \wedge \cdots \wedge d\hat{q}_{i_n}$ and using (3.12) we expand this as a polynomial in $\eta_{i_1}, ..., \eta_{i_n}$ (with differential forms as coefficients). It is a polynomial of degree $n$ with term of degree $n$,

$$\left(\prod_{k=1}^{n} \eta_{i_k}\right) d(\langle z, \alpha_{i_1}\rangle^D) \wedge \cdots \wedge d(\langle z, \alpha_{i_n}\rangle^D). \tag{3.19}$$

Using (3.13) we conclude there is an open dense set $G_1 \subset \mathbb{C}^n$ such that, if $(\eta_{i_1}, ..., \eta_{i_n}) \in G_1$, then $d\hat{q}_i \wedge \cdots \wedge d\hat{q}_{i_n} \not\equiv 0$ on $\mathbb{C}^n$. Now, by (3.14), $\hat{W}_j$ is a non-zero constant multiple of $\hat{q}_j$ (for $j=1, ..., s$) except for at most one value of $\eta_j$, so we may conclude there is an open dense set $G_2 \subset \mathbb{C}^s$ such that for $\eta \in G_2$ (3.10) is satisfied.

This completes the proof of Lemma 3.1.

*Proof* (*of Theorem* 3.2). Let $W_1, ..., W_s$ satisfy (3.8), (3.9), and (3.10) of Lemma 3.1. Consider, for $R_1, ..., R_s$ real and positive, the polynomial polyhedron

$$Y_R = Y(R_1, ..., R_s) = \{z \in \mathbb{C}^n \mid |\hat{W}_j(z)| < R_j \text{ for } j=1, ..., s\}. \tag{3.20}$$

$\bar{Y}_R$ is compact since (3.9) is satisfied. It is connected since each $\hat{W}_j$ is homogeneous, so $Y_R$ is, in fact, starlike with respect to 0.

Now, $Y(T_1, ..., T_s)$ is a Weil domain if for every integer $\ell$, $1 \leqslant \ell \leqslant n$, and every subset of cardinality $\ell$, $\{i_1, ..., i_\ell\} \subset \{1, ..., s\}$, then $(T_{i_1}^2, ..., T_{i_\ell}^2)$ is a regular value of the mapping

$$z \to (|\hat{W}_{i_1}|^2, ..., |\hat{W}_{i_\ell}|^2). \tag{3.21}$$

Equation (3.10) implies that $d\hat{W}_{i_1} \wedge \cdots \wedge d\hat{W}_{i_\ell} \not\equiv 0$ on $\mathbb{C}^n$ so that it follows from standard results in algebraic geometry (e.g., [11, Theorem 6, p. 50]) that the range of the map $z \to (\hat{W}_{i_1}, ..., \hat{W}_{i_\ell})$ is an open dense set in $\mathbb{C}^\ell$ (in fact, the complement of an algebraic variety). Thus, there is an open dense set $G \subset (\mathbb{R}^+)^s$ such that for $(T_1, ..., T_s) \in G$ then for every integer $\ell$, $1 \leqslant \ell \leqslant n$, and every subset $\{i_1, ..., i_l\} \subset \{1, ..., s\}$, $(T_{i_1}^2, ..., T_{i_\ell}^2)$ is in the range of the mapping given by (3.21). Thus given any $T > 0$, and $0 < \delta < T$, there are real numbers $T_1, ..., T_s$ satisfying

$$T - \delta < T_j < T + \delta \qquad \text{for} \quad j=1, ..., s \tag{3.22}$$

such that $Y(T_1, ..., T_s)$ is a Weil domain. Fix $T, T_1, ..., T_s$ sufficiently large and $\delta$ sufficiently small so that $Y(T_1, ..., T_s)$ is a Weil domain, (3.22) is satisfied, and $E \subset Y(T_1, ..., T_s)$. $Y(T_1, ..., T_s)$ will be denoted simply by $Y$. We will apply Weil's integral formula to $\hat{Q}_d(z)$. (For Weil's integral formula and related notions we will use the notation of [10].)

We obtain an expansion [10, Theorem 2, p. 165]

$$\hat{Q}_d = \sum_{|k|=0}^{\infty} \sum_I{}' A_k^I(z)(\hat{W}_I(z))^k \tag{3.23}$$

where $k=(k_1, ..., k_n)$ and $I=(i_1, ..., i_n)$ are multi-indices. The inner sum is over multi-indices $I=(i_1, ..., i_n)$ with $1 \leqslant i_1 < \cdots < i_n \leqslant s$. The coefficients $A_k^I(z)$ are given by

$$A_k^I(z) = \frac{1}{(2\pi i)^n} \int_{\sigma_I} \frac{\hat{Q}_d(\zeta)\, K_I(z, \zeta)\, d\zeta}{(\hat{W}_{i_1}(\zeta))^{k_1+1} \cdots (\hat{W}_{i_n}(\zeta))^{k_n+1}}, \tag{3.24}$$

where $\sigma_I$ is a suitably oriented edge of the Weil domain, $Y(T_1, ..., T_s)$, and

$$K_I(z, \zeta) = \det(P_j^{i_l}) \qquad j, l = 1, ..., n, \tag{3.25}$$

and the functions $P_v^i$ come from the Hefer expansion

$$\hat{W}_i(\zeta) - \hat{W}_i(z) = \sum_{v=1}^{n} (\zeta_v - z_v)\, P_v^i(\zeta, z). \tag{3.26}$$

We will take

$$P_v^i(\zeta, z) = \frac{\hat{W}_i(z_1 \cdots z_{v-1}, \zeta_v \cdots \zeta_n) - \hat{W}_i(z_1, \cdots z_v, \zeta_{v+1} \cdots \zeta_n)}{\zeta_v - z_v}. \tag{3.27}$$

Since each $\hat{W}_i$ is homogeneous of degree $D$, $K_I(z, \zeta)$ is homogeneous in $(\zeta, z)$ of degree $n(D-1)$. Thus $K_I(z, \zeta)$ and $A_k^I(z)$ are considered as polynomials in $z$ of degree $\leqslant n(D-1)$. We will write them as a sum of homogeneous polynomials in $z$

$$K_I(z, \zeta) = \sum_{r=0}^{n(D-1)} K_{I, r}(z, \zeta) \tag{3.28}$$

and

$$A_k^I(z) = \sum_{r=0}^{n(D-1)} A_{k, r}^I(z). \tag{3.29}$$

Of course, $A_{k, r}^I(z)$ is given by the integral (3.24) where $K_i(z, \zeta)$ is replaced by $K_{I, r}(z, \zeta)$.

Now $\hat{Q}_d(z)$ is homogeneous of degree $d$ and so equating terms of the same homogeneity in (3.23) we have

$$\hat{Q}_d(z) = \sum_{\substack{k \\ r + D\,|k| = d}} \sum_I{}' \sum_{r=0}^{n(D-1)} A_{k, r}^I(z) (\hat{W}_I(z))^k. \tag{3.30}$$

This is a finite sum. Let

$$H_d(z) = \sum_{\substack{k \\ r + D\,|k| = d}} \sum_{I}{}' \sum_{r=0}^{n(D-1)} A^I_{k,r}(z)(W_I(z))^k. \tag{3.31}$$

That is, $H_d(z)$ is obtained from the expression (3.30) for $\hat{Q}_d(z)$ by replacing $\hat{W}_I$ by $W_I$. $H_d(z)$ is polynomial of degree $d$ and $\hat{H}_d(z) = \hat{Q}_d(z)$. Thus, by definition, $\|\mathrm{Tch}_E\, \hat{Q}_d(z)\|_E \leqslant \|H_d(z)\|_E$ and we will now estimate $\|H_d(z)\|_E$.
Using (3.8) we have

$$\|H_d\|_E \leqslant \sum_{\substack{k \\ r + |k|\,D = d}} \sum_{I}{}' \sum_{r=0}^{n(D-1)} \|A^I_{k,r}(z)\|_E. \tag{3.32}$$

We will use (3.24) to estimate $\|A^I_{k,r}\|_E$. Using (3.9) and the hypothesis of Theorem 3.2 we have

$$\overline{\lim_{d\to\infty}}\, \frac{1}{d} \log |\hat{Q}_d(z)| \leqslant \operatorname*{Max}_{1\leqslant j\leqslant s} \left(\frac{1}{D} \log |\hat{W}_j(z)|\right) + \varepsilon$$

for all $z \in \mathbb{C}^n - \{0\}$. Hence, for $z \in Y$, and using (3.22),

$$\overline{\lim_{d\to\infty}}\, \frac{1}{d} \log |\hat{Q}_d(z)| \leqslant \log \frac{(T+\delta)}{D} + \varepsilon$$

and then using Hartogs' lemma and exponentiating,

$$\overline{\lim_{d\to\infty}}\, \|\hat{Q}_d\|_Y \leqslant (T+\delta)^{d/D}\, e^{\varepsilon d}. \tag{3.33}$$

Then, since $E \subset Y$ and each $\sigma_I$ has finite volume, there is a constant $C > 0$ such that for all $f$ continuous on $\overline{Y}$ and all $I, r$

$$\left\| \left(\frac{1}{2\pi}\right)^n \int_{\sigma_I} f(\zeta)\, K_{I,r}(z,\zeta)\, d\zeta \right\|_E \leqslant C\, \|f\|_{\overline{Y}}. \tag{3.34}$$

Thus we have

$$\|A^I_{k,r}\|_E \leqslant C(T-\delta)^{-(|k|+n)}\, (T+\delta)^{d/D}\, e^{\varepsilon d}.$$

Using (3.32) and the fact that the number of terms in the sum on the right of (3.32) is bounded by a polynomial in $d$ we have

$$\|H_d\|_E \leqslant C\,(\text{polynomial in } d)(T-\delta)^{-(|k|+n)}\, (T+\delta)^{d/D}\, e^{\varepsilon d}. \tag{3.35}$$

For non-zero terms in (3.32) we must have

$$\frac{d-n(D-1)}{D} \leqslant |k| \leqslant \frac{d}{D}. \tag{3.36}$$

As $d \to \infty$, $|k|/d \to 1/D$ for all $k$ satisfying (3.36). Hence,

$$\varlimsup_{d \to \infty} \|\mathrm{Tch}\, \hat{Q}_d\|_E^{1/d} \leqslant \varlimsup_{d \to \infty} \|H_d\|_E^{1/d} \leqslant \left(\frac{T+\delta}{T-\delta}\right) e^{\varepsilon}. \tag{3.37}$$

But $\delta, \varepsilon > 0$ are arbitrary so Theorem 3.2 follows.

COROLLARY 3.1. *Let $f \in W(E)$ and let $\{B_d\}_{d=1,2,\ldots}$ be a sequence of best approximants to $f$ from $\mathscr{P}_d^n$. Suppose that, for some $\xi_0 \in \mathbb{C}^n - \{0\}$,*

$$\varlimsup_{d \to \infty} \frac{1}{d} \log |\beta_d(\xi_0)| - \log |\xi_0| = \rho_E([\xi_0]).$$

*Then $f$ is not analytic on $E$ (i.e., in a neighborhood of $E$).*

*Remark* 3.1. In the case $n=1$, the above condition is necessary and sufficient for $f \in W(E)$ not to be analytic on $E$ [2, Theorem 2.1].

Let $d_r(E) = \sup_{|\alpha|=r} \|\mathrm{Tch}_E\, z^\alpha\|_E$ and let $d(E) = \lim_{r \to \infty} (d_r(E))^{1/r}$. Szczepański [14] proved the following result.

THEOREM 3.3. *Let $f \in W(E)$ and let $\{B_r\}_{r=1,2,3,\ldots}$ be a sequence of polynomials, best approximants to $f$ from $\mathscr{P}_r^n$. Suppose $\beta_r = \sum_{|\alpha|=r} a_{r,\alpha} z^\alpha$, and that, for some $R > 1$,*

$$\varlimsup_{r \to \infty} \left(\sum_{|\alpha|=r} |a_{r,\alpha}|\right)^{1/r} = \frac{1}{Rd(E)}. \tag{3.38}$$

*Then $f$ extends to be holomorphic on $E_R$.*

This result follows from Theorem 3.1, since

$$\begin{aligned}\|\mathrm{Tch}_E\, \beta_r\|_E &\leqslant \sum_{|\alpha|=r} |a_{r,\alpha}|\, \|\mathrm{Tch}_E\, z^\alpha\|_E \\ &\leqslant \sum_{|\alpha|=r} |a_{r,\alpha}|\, (d_r(E)).\end{aligned}$$

Hence if (3.34) is satisfied we have $\overline{\lim}_{r\to\infty} \|\mathrm{Tch}_E\, \beta_r\|_E^{1/r} \leqslant 1/R$ and condition (3.4) of Theorem 3.1 is satisfied.

Szczepański's necessary condition [14, Theorem 2.6] for $f \in W(E)$ to extend holomorphically to $E_R$ is that, for any sequence $\{B_d\}$ of best approximants to $f$ from $\mathscr{P}_d^n$, we have

$$\overline{\lim_{d\to\infty}}\ \|\beta_d\|_{\Delta}^{1/d} \leqslant \left(\frac{1}{RC_m(E)}\right) \tag{3.39}$$

where $\Delta$ is the unit polydisc in $\mathbb{C}^n$ and

$$\log\left(\frac{1}{C_m(E)}\right) = \overline{\lim_{r\to\infty}}\ \{\sup_{z\in\Delta} V_E(rz) - \log r\}.$$

We will show that (3.39) can be deduced from (3.5).

Now, if (3.5) of Theorem 3.1 holds and using the definition of Robin function, we have, for all $z \in \mathbb{C}^n - \{0\}$,

$$\overline{\lim}\ \frac{1}{d} \log |\beta_d(z)| \leqslant \overline{\lim_{\substack{|\lambda|\to\infty \\ \lambda\in\mathbb{C}}}}\ V_E(\lambda z) - \log|\lambda| - \log R. \tag{3.40}$$

If we restrict $z$ to $\Delta$, the right-hand side of (3.40) is

$$\leqslant \overline{\lim_{r\to\infty}}\ \{\sup_{z\in\Delta} V_E(rz) - \log r\} - \log R.$$

Taking the sup of $z \in \Delta$ of the left-hand side of (3.40), we have

$$\overline{\lim_{d\to\infty}}\ \frac{1}{d} \log \|\beta_d(z)\|_{\Delta} \leqslant \overline{\lim_{r\to\infty}}\ \{\sup_{z\in\Delta} V_E(rz) - \log r\} - \log R$$

which, taking exponentials, gives (3.39).

EXAMPLE 3.1. We will give an example of a function $f$ where (3.4) is satisfied for some $R > 1$ but (3.38) is not satisfied for any $R > 1$. We use the set $K_\gamma$ of Example 2.1.

For $m_1$, $m_2$, $r$ integers $\geqslant 0$, we have (see Example 2.1)

$$\mathrm{Tch}_{K_\gamma} z_1^{m_1} z_2^{m_2} = z_1^{m_1} z_2^{m_2} \qquad \text{and} \quad \mathrm{Tch}_{K_\gamma}(z_1 - z_2)^r = (z_1 - z_2)^r. \tag{3.41}$$

Thus using the notation of Theorem 3.3 we have $d_r(K_\gamma)=1$ for all $r=1,2,3,\ldots$ so $d(K_\gamma)=1$. Let $f(z_1,z_2)=1/(2\gamma-(z_1-z_2))$ and let $B_r(t)=b_r t^r+(\text{lower order terms})$ be the best approximant from $\mathscr{P}^1_d$ to $1/(2\gamma-t)$ on $|t|\leqslant\gamma$. Then

$$\overline{\lim_{r\to\infty}}\ |b_r|^{1/r}=\frac{1}{2\gamma}. \tag{3.42}$$

$B_r(z_1-z_2)$ is a best approximant in $\mathscr{P}^2_r$ to $f$ on $K_\gamma$ as may easily be deduced by considering the linear automorphism of $\mathbb{C}^2=(z_1,z_2)\to(z_1-z_2,z_1)$ and $\beta_r=b_r(z_1-z_2)^r$. Using (3.41) and (3.42) we have

$$\overline{\lim_{r}}\ \|\mathrm{Tch}_{K_\gamma}\beta_r\|^{1/r}=\tfrac{1}{2}, \tag{3.43}$$

and (3.4) is satisfied for $R=2$. But $\beta_r=\sum_{|\alpha|=r}a_{r,\alpha}z^\alpha$ and

$$\sum_{|\alpha|=r}|a_{r,\alpha}|=b_r2^r \qquad \text{so} \quad \lim_{r\to\infty}\Bigg(\sum_{|\alpha|=r}|a_{r,\alpha}|\Bigg)^{1/r}=\frac{1}{\gamma}>\frac{1}{d(K_\gamma)}, \tag{3.44}$$

so (3.38) is not satisfied for any $R>1$.

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